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Kergin interpolation in Banach spaces

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Abstract

We show that Kergin interpolation, a generalized Lagrange–Hermite polynomial interpolation, may be defined on mappings between general Banach spaces. Like its finite-dimensional counterpart, Kergin interpolation in this setting is an affine-invariant projector. We obtain an error formula which we use to approximate holomorphic mappings. As an application we give a convergence theorem applicable to, for instance, operators on Banach algebras, such as the algebra of square matrices with complex coefficients.

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1. Introduction

Kergin interpolation is a generalization to several variables of ordinary Lagrange–Hermite interpolation, cf. [9–11] for the real case and [1,2] for complex Kergin interpolation, see also [6] where more general mean-value interpolation is studied. The aim of this paper is to generalize Kergin interpolation to mappings between Banach spaces.

Recall that one-variable Lagrange interpolation is not merely a matter of matching function values at certain points, it also enjoys several analytical properties. Most importantly, Gelfond [7] proved a number of convergence theorems for sequences of interpolating polynomials, giving conditions on an

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infinite sequence of points and on an entire function f ensuring that the sequence of successive Lagrange polynomials at the first n points converges to f uniformly on compact sets as n tends to infinity. Similar theorems hold for Kergin interpolation in $\mathbb{C}^n$, due to Bloom [4], and this is why it is justifiable to look upon Kergin interpolation as the "correct" generalization to several variables of one-variable Lagrange interpolation. It is certainly of interest to obtain such a construction in general Banach spaces, and we present a convergence result of Gelfond's type for the infinite-dimensional Kergin interpolation that we study.

The main tool for defining Kergin interpolation in $\mathbb{R}^n$ is the simplex functional, which, for a given sequence of points $(p_0, p_1, \ldots, p_j)$ in $\mathbb{R}^n$ is defined for, say, functions continuous on the convex hull of the points by

$$g \mapsto \int_{[p_0, p_1, \ldots, p_j]} g := \int_{S_j} g(p_0 + s_1(p_1 - p_0) + \cdots + s_j(p_j - p_0))\, ds_1 \cdots ds_j,$$

where $S_j = \{(s_1, \ldots, s_j)\colon\ s_i \geqslant 0,\ \sum s_i \leqslant 1\}$ is the standard j-simplex in $\mathbb{R}^j$. Let D_y denote the directional derivative along the vector y. Then the Kergin polynomial of a sufficiently smooth f with respect to a sequence of points $p = (p_0, p_1, \ldots, p_k)$ is

$$K_p f(x) = f(p_0) + \int_{[p_0, p_1]} D_{x-p_0} f + \cdots + \int_{[p_0, p_1, \ldots, p_k]} D_{x-p_{k-1}} \cdots D_{x-p_0} f.$$

This definition is also valid in the complex setting for functions holomorphic on the convex hull of the given points. The Kergin polynomial is the unique polynomial of degree at most k satisfying

$$\int_{[p_0, p_1, \ldots, p_j]} D^{\alpha}(f - K_p f) = 0, \quad |\alpha| = j, \quad j = 0, 1, \ldots, k.$$

The Kergin polynomial $K_p f$ matches the values of f at each of the p_i. Also, the operator K_p taking a function to its Kergin polynomial preserves polynomials of degree at most k and is affine-invariant, in the sense that for every affine map $A : \mathbb{R}^n \to \mathbb{R}^m$ and any suitably defined function g we have

$$K_p(g \circ A) = (K_{Ap} g) \circ A,$$

where $Ap := (A(p_0), A(p_1), \ldots, A(p_k))$. Note that in dimension one we have Genocchi's formula

$$[p_0, p_1, \ldots, p_j]f = \int_{[p_0, p_1, \ldots, p_j]} f^{(j)},$$

and so the Kergin polynomial in this case is nothing but the Lagrange polynomial in Newton form (the entity on the left-hand side is, of course, the jth divided difference of the function f).

To extend the definition of Kergin interpolation to mappings between general Banach spaces we need to replace the Lebesgue integral by a Bochner integral to get a vector valued version of the simplex functional (this is then no longer a functional in the usual sense of the word). The generalized Kergin polynomial will then have the same form as above. The main properties of classical Kergin interpolation, including

Micchelli's error formula, are easily extended to this setting. They are stated in Theorems 5.7 and 6.1.

Using the error formula we prove an approximation theorem (Theorem 6.2) which says that, under certain conditions, the Kergin interpolants of a holomorphic function converge geometrically fast to the function.

We shall also study Kergin interpolation of entire power series from one Banach space into another. We obtain a precise result (Theorem 7.1) giving conditions under which a sequence of Kergin polynomials of an entire power series f converges to f uniformly on every bounded ball. As was mentioned above, this generalizes results of Gelfond [7] and Bloom [4]. Our method is different.

We shall start out by listing some standard properties of polynomials in Banach spaces, holomorphic mappings in Banach spaces and vector valued integration. All facts presented here are well known, but we include them for completeness and ease of reference. Also, these introductory sections serve the purpose of fixing the notation. The reader should observe that we use the same notation ($\|\cdot\|$) to denote all the (different) norms of vectors, linear maps and multilinear maps.

2. Polynomials in Banach spaces

Throughout this paper, X and Y will denote complex (or real) Banach spaces. For each positive integer k we let $\mathscr{L}_k(X,Y)$ denote the set of continuous k-linear mappings from X^k into Y, i.e. mappings $T: X^k \to Y$ that are linear in each argument separately. It is an elementary fact that this is again a Banach space when equipped with the standard norm

$$\|T\| := \sup_{x_1 \neq 0, \ldots, x_k \neq 0} \frac{\|T(x_1, \ldots, x_k)\|}{\|x_1\| \cdots \|x_k\|},$$

where $x_1, \ldots, x_k \in X$. Note that

$$\|T(x_1, \ldots, x_k)\| \leqslant \|T\| \, \|x_1\| \cdots \|x_k\|.$$

If $T \in \mathscr{L}_k(X,Y)$ and $x_1 = x_2 = \cdots = x_k = x$, then we will often write

$$T(x_1, \ldots, x_k) = Tx^k.$$

A mapping $H: X \to Y$ is said to be a continuous homogeneous polynomial of degree k if there exists an $L \in \mathscr{L}_k(X,Y)$ such that

$$H(x) = Lx^k, \quad x \in X.$$

Given such a homogeneous polynomial H of degree k there exists a unique *symmetric* (i.e. invariant under permutation of the variables) continuous k-linear form B such that

$$H(x) = Bx^k.$$

This symmetric k-linear form is called the *polar form* of H.

Now a mapping $P : X \to Y$ is called a polynomial if it is a sum of homogeneous polynomials. To be precise, P is a continuous polynomial of degree d if there exist mappings $L_0, L_1, \dots, L_d$ such that $L_k \in \mathscr{L}_k(X, Y)$ for $k = 0, 1, \dots, d$, $L_d \neq 0$, and for all $x \in X$,

$$P(x) = L_0 + L_1 x + L_2 x^2 + \cdots + L_d x^d.$$

Let f be a mapping from an open subset U of X into Y. If for each point $a \in U$ there exists a linear mapping $T : X \to Y$ such that

$$\lim_{x \to a} \frac{||f(x) - f(a) - T(x-a)||}{||x - a||} = 0,$$

then f is said to be differentiable on U. The map T is uniquely determined by f and a, it is called the differential of f at a and will be denoted by $Df(a)$. Hence a continuously differentiable function $f : U \to Y$ induces a mapping $Df : X \to \mathscr{L}_1(X, Y)$. If, for instance, $f(x) = Bx^j$ where B is a symmetric j-linear map, then

$$Df(x) = B(\cdot, x, \dots, x) + B(x, \cdot, x, \dots, x) + \cdots + B(x, \dots, x, \cdot) = jB(x, \dots, x, \cdot)$$

with x appearing $j - 1$ times inside the parentheses.

Higher order differentials are defined in the obvious way, e.g. the second-order differential of f at a is the differential of Df at a and is denoted by $D^2 f(a)$. Note that for each k, $\mathscr{L}_k(X, Y)$ is isometric to $\mathscr{L}_1(X, \mathscr{L}_{k-1}(X, Y))$, which in turn is isometric to $\mathscr{L}_1(X, \mathscr{L}_1(X, \mathscr{L}_{k-2}(X, Y)))$, and so on. Hence $D^k f(a)$ may be regarded as a member of $\mathscr{L}_k(X, Y)$, and it is a classical theorem that $D^k f(a)$ is then a symmetric k-linear form.

As usual we let $C^k(U, Y)$ denote the vector space of all k times continuously differentiable maps from U into Y. This space is endowed with the topology of uniform convergence on compact sets of the mappings and their differentials up to order k, i.e. the topology generated by the seminorms

$$f \mapsto \sup_{j \leqslant k} \sup_{x \in K} ||D^j f(x)||,$$

where K is any compact subset of U. This is often referred to as the C^k topology.

We will also have use for directional derivatives, given by

$$D_y f(x) = Df(x)(y),$$

i.e. the derivative of f at x in the direction y is the differential of f at x applied to the vector y.

Throughout this paper, we will denote by $B_r(\xi)$ the open ball of radius r centered at ξ. Similarly, by $\overline{B_r(\xi)}$ we denote the closed ball of radius r centered at ξ.

3. Holomorphic mappings between Banach spaces

Let X and Y be complex Banach spaces and U an open subset of X. A mapping $f : U \to Y$ is said to be holomorphic on U if for every $\xi \in U$ there exist an open ball $B_r(\xi) \subset U$ and a sequence of continuous j-linear mappings $L_j \in \mathscr{L}_j(X, Y)$ such that

$$f(x) = \sum_{j=0}^{\infty} L_j(x - \xi)^j,$$

uniformly for $x \in B_r(\xi)$. The sequence L_j is then unique, and in fact $L_j = D^j f(\xi)/j!$. In view of the close relation between j-linear mappings and homogeneous polynomials of degree j, it is of course equivalent to say that f is holomorphic if there exists a sequence of continuous polynomials P_j such that each P_j is homogeneous of degree j and

$$f(x) = \sum_{j=0}^{\infty} P_j(x - \xi),$$

uniformly for $x \in B_r(\xi)$.

Variants of the Cauchy integrals are valid, and they give rise to estimates on the size of the coefficients in the power series.

Proposition 3.1. *Let X and Y be complex Banach spaces, and let U be an open subset of X. If $f : U \to Y$ is holomorphic and $\overline{B_r(\xi)} \subset U$ for some $r > 0$, then, for each $j \geqslant 0$,*

$$||D^j f(\xi)|| \leqslant j^j \left(\sup_{||x-\xi||=r} ||f(x)|| \right) r^{-j}.$$

Proof. See Proposition 3 of Chapter 6 in [13] or [12]. $\square$

4. Vector valued integration

Let (X, Σ, μ) be a finite measure space and let Y be a Banach space. A mapping $f : X \to Y$ is said to be simple if there exist disjoint sets $E_1, \dots, E_k$ in Σ and vectors $y_1, \dots, y_k$ in Y such that, for all $x \in X$,

$$f(x) = \sum_{j=1}^{k} \chi_j(x) y_j,$$

where χ_j is the characteristic function of E_j. The Bochner integral of such an f over any set E in Σ is then defined to be

$$\int_E f \, d\mu = \sum_{j=1}^{k} \mu(E \cap E_j) y_j.$$

A mapping $f: X \to Y$ is said to be measurable if there exists a sequence of simple mappings f_n that converges to f almost everywhere. A measurable mapping $f: X \to Y$ is said to be Bochner integrable if there exists a sequence of simple mappings f_n such that

$$\lim_{n\to\infty} \int_X ||f_n - f||\, d\mu = 0.$$

Then the Bochner integral of f over any E in Σ is defined by

$$\int_E f\, d\mu = \lim_{n\to\infty} \int_E f_n\, d\mu.$$

This is well-defined, and it turns out that, for example, all continuous mappings are Bochner integrable. Also, analogues of the standard Lebesgue theorems hold in this setting, cf. [5]. The following proposition is useful.

Proposition 4.1. *Let (X, Σ, μ) be a finite measure space and $f: X \to Y$ Bochner integrable. Then the following holds*:

(1) *For each continuous linear functional ψ in the dual space X^*, the function $\psi\circ f$ is integrable and for all E in Σ*

$$\psi\left(\int_E f\, d\mu\right) = \int_E \psi\circ f\, d\mu.$$

(2) *The function $||f||$: $X \to \mathbb{R}$ is integrable and for all E in Σ*

$$\left|\left|\int_E f\, d\mu\right|\right| \leqslant \int_E ||f||\, d\mu.$$

Proof. See Proposition 6.4 in [12]. $\square$

5. Kergin interpolation in Banach spaces

In the finite-dimensional case, the coefficients of the Kergin polynomial are (linear combinations of) so-called simplex functionals. Let us first generalize this concept to the Banach space setting.

Definition 5.1. Let $p = (p_0, p_1, \ldots, p_j)$ be a sequence of points in X. The simplex functional with respect to p is defined for any function g continuous on the convex hull of p by

$$g \mapsto \int_{[p_0, p_1, \ldots, p_j]} g := \int_{S_j} g(p_0 + s_1(p_1 - p_0) + \cdots + s_j(p_j - p_0))\, ds_1 \cdots ds_j,$$

where $S_j = \{(s_1, \ldots, s_j)\colon\ s_i \geqslant 0,\ \sum s_i \leqslant 1\}$ is the standard j-simplex in $\mathbb{R}^j$.

Remark 5.2. The integral on the right in Definition 5.1 is a Bochner integral with respect to Lebesgue measure. Hence our simplex functional takes values in a Banach space and is not a functional in the usual sense.

Remark 5.3. When g is scalar-valued, many properties of the finite-dimensional simplex functional immediately generalize to the infinite-dimensional setting, since everything takes place inside the finite-dimensional affine subspace spanned by the points in the sequence p. To be precise, let $\mathbf{P}$ be the vector space spanned by p. Then

$$\int_{[p]} g = \int_{[p]} g|_{\mathbf{P}},$$

where on the right-hand side everything lives in the finite-dimensional space $\mathbf{P}$ ($g|_{\mathbf{P}}$ denotes the restriction of g to $\mathbf{P}$). This means that all the known results about the simplex functional on finite-dimensional spaces extend immediately to scalar-valued functions on infinite-dimensional Banach spaces.

Proposition 5.4. *Let $p = (p_0, p_1, \ldots, p_j)$ be a sequence of vectors in the Banach space X and let Ω be the convex hull of p. The simplex functional defined above has the following properties*:

(1) *For every continuous mapping $f : \Omega \to Y$, the vector $\int_{[p]} g$ is independent of the ordering of the points in p.*
(2) *It is affine-invariant, in the sense that if A is a continuous affine map of X into another Banach space Z, and $g : A(\Omega) \to Y$ is continuous, then*
$$\int_{[p]} g \circ A = \int_{[Ap]} g,$$
where $Ap = (A(p_0), A(p_1), \ldots, A(p_j))$.

Proof. Using Proposition 4.1 and the affine invariance in the finite-dimensional case (and Remark 5.3 above), we get, for each continuous linear functional ψ on Y,

$$\psi\left(\int_{[Ap]} g\right) = \int_{[Ap]} \psi \circ g = \int_{[p]} \psi \circ g \circ A = \psi\left(\int_{[p]} g \circ A\right).$$

Since the functionals separate points, (2) follows. Property (1) follows in a similar way from the corresponding property in the finite-dimensional case and Proposition 4.1. $\square$

Now we can define the Kergin polynomial associated to a mapping f with respect to a sequence p of vectors.

Definition 5.5. Let X and Y be complex (or real) Banach spaces and let U be a convex open subset of X. If $p = (p_0, p_1, \ldots, p_k)$ is a sequence of vectors in U and

$f \in C^k(U, Y)$, then the Kergin polynomial of f with respect to p is defined by

$$K_p f(x) = f(p_0) + \int_{[p_0, p_1]} D_{x-p_0} f + \cdots + \int_{[p_0, p_1, \dots, p_k]} D_{x-p_{k-1}} \cdots D_{x-p_0} f.$$

The following proposition shows that we can essentially restrict our study to scalar-valued functions.

Proposition 5.6. *Let $K_p f$ be as in Definition* 5.5. *For every continuous linear functional ψ on Y, we have for all x,*

$$\psi(K_p f(x)) = K_p(\psi \circ f)(x). \tag{5.1}$$

Proof. With the above notation, we have

$$\begin{aligned}\psi\left(\int_{[p_0, \dots, p_i]} D_{x-p_{i-1}} \cdots D_{x-p_0} f\right) &= \int_{[p_0, \dots, p_i]} \psi(D_{x-p_{i-1}} \cdots D_{x-p_0} f) \\ &= \int_{[p_0, \dots, p_i]} D_{x-p_{i-1}} \cdots D_{x-p_0}(\psi \circ f).\end{aligned}$$

Summing up for $i = 0, \dots, k$ we get the desired formula. □

On the other hand, if **V** is a finite-dimensional subspace of X that contains all of the p_i together with x, then we have

$$K_p f(x) = K_p(f|_{\mathbf{V}})(x). \tag{5.2}$$

Combining (5.1) and (5.2), we get

$$\psi(K_p f(x)) = K_p(\psi \circ f|_{\mathbf{V}})(x), \tag{5.3}$$

where the Kergin operator on the right-hand side is the classical one. This leads to an immediate extension of basic properties of classical Kergin interpolation. The following theorem states these properties.

Theorem 5.7. *Let X and Y be complex* (*or real*) *Banach spaces and let U be a convex open subset of X. If $f \in C^k(U, Y)$ and $p = (p_0, p_1, \dots, p_k)$ is a sequence of vectors in U, then the mapping $K_p f$ defined above is a polynomial of degree at most k such that*

$$K_p f(p_j) = f(p_j), \quad j = 0, 1, \dots, k. \tag{5.4}$$

Moreover, the operator $K_p : C^k(U, Y) \to C^k(U, Y)$ taking a function to its Kergin polynomial is continuous in the C^k topology, and it has the following properties:

(1) *It is independent of the ordering of the points in the sequence p.*
(2) *It is associative, i.e. if $p \subset q$ then $K_p f = K_p K_q f$.*
(3) *It is affine-invariant, in the sense that if $f = g \circ A$, where A is a continuous affine map from X into a Banach space Z and $g \in C^k(A(U), Y)$, then*

$$K_p f = K_p(g \circ A) = (K_{Ap} g) \circ A.$$

(4) *It is a projector, i.e. if f already is a polynomial of degree at most k, then*

$$K_p f(x) = f(x), \quad x \in U.$$

Proof. To show that $K_p f$ is a polynomial, it is enough to show that each of the terms in the sum defining $K_p f$ is a polynomial, and this is immediate from the definitions. The degree is obviously at most k.

The continuity of the operator K_p is easily deduced from Proposition 5.6, using the seminorms defining the C^k topology and the fact that the convex hull of a finite number of points is a compact subset of U.

We prove that $K_p f$ has the interpolating property (5.4). Let $\mathbf{V}$ be a finite-dimensional subspace that contains p. For every continuous linear functional ψ on Y we have, in view of (5.3),

$$\psi(K_p f(p_j)) = K_p(\psi \circ f|_{\mathbf{V}})(p_j) = \psi(f(p_j)).$$

Here the second equality follows from the classical result. Now since functionals separate points we deduce that $K_p f(p_j) = f(p_j)$.

Properties (1)–(4) follow easily in the same way from similar properties in the finite-dimensional case. Let us just explain the last one. Let q be a degree k polynomial. We shall prove that for all $x \in X$, $K_p q(x) = q(x)$. Fix x and take $\mathbf{V}$ to be a finite-dimensional subspace of X that contains p and x. For every ψ as above, $\psi \circ q|_{\mathbf{V}}$ is a classical polynomial of degree at most k, and consequently $K_p(\psi \circ q|_{\mathbf{V}})(x) = (\psi \circ q|_{\mathbf{V}})(x) = \psi(q(x))$. It follows that $K_p q(x) = q(x)$. $\square$

Remark 5.8. A result similar to Theorem 5.7 holds for holomorphic mappings defined in so-called $\mathbb{C}$-convex domains, see [3,6] or [8] for the definition and further properties of such domains. This will be treated in a forthcoming paper.

6. An error formula

This section is devoted to obtaining an error formula for Banach space Kergin interpolation. Our formula is a generalization of Micchelli's formula in [10], which in turn is a multi-variate analogue of the formula

$$g(t) - L_p g(t) = [p_0, p_1, \ldots, p_k, t]g,$$

valid for the Lagrange polynomial $L_p g$ of a one-variable function g. We will prove the following.

Theorem 6.1. *Let X and Y be complex (or real) Banach spaces and let U be a convex open subset of X. If $f \in C^k(U, Y)$ and $p = (p_0, p_1, \ldots, p_k)$ is a sequence of vectors in U,* *then, for each* $x \in U$,

$$f(x) - K_p f(x) = \int_{[p_0, p_1, \dots, p_k, x]} D_{x-p_k} \cdots D_{x-p_0} f. \tag{6.1}$$

Proof. This is again a consequence of the classical formula via (5.3). Fix $x \in U$ and let $\mathbf{V}$ be a finite-dimensional subspace that contains p and x. For every continuous linear functional ψ we have, using the classical result at the second equality,

$$\begin{aligned}\psi(f(x) - K_p f(x)) &= (\psi \circ f|_{\mathbf{V}})(x) - K_p(\psi \circ f|_{\mathbf{V}})(x) \\ &= \int_{[p_0, \dots, p_k, x]} D_{x-p_k} \cdots D_{x-p_0}(\psi \circ f|_{\mathbf{V}}) \\ &= \psi\left(\int_{[p_0, \dots, p_k, x]} D_{x-p_k} \cdots D_{x-p_0} f\right).\end{aligned}$$

We reach the desired conclusion by using once more the fact that the functionals separate points. □

In one dimension, it is a classical result that the Lagrange interpolants of a function, analytic in a sufficiently large region containing the points of interpolation, converge geometrically fast to the function as the number of points increases. The corresponding result for Kergin interpolation was proved by Micchelli [10]. In the infinite-dimensional case we have the following:

Theorem 6.2. *Let* X *and* Y *be complex Banach spaces and let* $p = (p_0, p_1, p_2, \dots)$ *be an infinite sequence of points in the closed unit ball* $B = \{x \in X\colon \|x\| \leqslant 1\}$. *Further, let* f *be a mapping into* Y, *holomorphic on the ball* $B' = \{x \in X\colon \|x\| \leqslant 2r + 1\}$, *where* $r > e$. *If* p^j *denotes the finite subsequence* $(p_0, p_1, \dots, p_j)$ *and* B'' *is any closed ball in* X *centered at the origin with radius* $2s + 1$, *where* $r > s > e$, *then*

$$\sup_{x \in B} \|f(x) - K_{p^j} f(x)\| \leqslant \sup_{x \in B''} \|f(x)\| \left(\frac{e}{s}\right)^{j+1}.$$

Hence, if f *is bounded on* B'', *the sequence of Kergin interpolants* $K_{p^j} f$ *converges geometrically fast to* f *on* B.

Proof. To begin with, we note that

$$D_{x-p_j} \cdots D_{x-p_0} f = D^{j+1} f(\cdot)(x - p_j, \dots, x - p_1, x - p_0).$$

By the remainder formula of Theorem 6.1 and Proposition 4.1, we obtain for each $x \in B$,

$$
\begin{aligned}
&||f(x) - K_{p^j} f(x)|| \\
&\leqslant \int_{[p_0, p_1, \ldots, p_j, x]} ||D_{x-p_j} \cdots D_{x-p_0} f|| \\
&\leqslant \int_{S_{j+1}} \left(\sup_B ||D^{j+1} f|| \right) ||x - p_0|| \, ||x - p_1|| \cdots ||x - p_j|| \, ds_1 \cdots ds_j.
\end{aligned}
$$

The volume of the simplex over which we integrate is $1/(j+1)!$. Further, by Proposition 3.1,

$$
\sup_B ||D^{j+1} f|| \leqslant \frac{(j+1)^{j+1}}{(2s)^{j+1}} \sup_{B''} ||f||.
$$

Since $||x - p_i|| \leqslant 2$ for each i and $(j+1)^{j+1}/(j+1)! < e^{j+1}$, we get

$$
||f(x) - K_{p^j} f(x)|| \leqslant \sup_{x \in B''} ||f(x)|| \left(\frac{e}{s} \right)^{j+1},
$$

for each $x \in B$. $\square$

Remark 6.3. We point out that, even though f is holomorphic on the ball B', there is no guarantee that f is bounded on the smaller ball B'', cf. [12, Proposition 7.15]. To ensure convergence in general, the assumption on the boundedness of f is needed. We also want to point out that Theorem 6.2 does not reduce to the known result in the finite-dimensional case, where it is enough to assume that $r > s > 1$.

7. Applications to entire power series

In this section, we prove a convergence theorem for entire power series from a Banach space X to a Banach space Y. What we mean by an entire power series is a mapping

$$
f(x) = \sum_{j=0}^{\infty} P_j(x),
$$

where each P_j is a continuous polynomial homogeneous of degree j from X into Y, such that, setting

$$
M_j := \sup_{||x||=1} ||P_j(x)||,
$$

we have

$$
\limsup_{j \to \infty} M_j^{1/j} = 0.
$$

This is equivalent to saying that

$$\limsup_{j \to \infty} c_j^{1/j} = 0,$$

where

$$c_j := ||B_j||$$

and B_j is the polar form of P_j. Indeed, with these notations, we always have

$$M_j \leqslant c_j \leqslant \frac{j^j}{j!} M_j.$$

Consequently, f is an entire power series when

$$\hat{f}(z) := \sum_{j=0}^{\infty} c_j z^j$$

is an entire function of one complex variable. Under these conditions f is infinitely differentiable and $D^k f$ is itself an entire power series from X into $\mathscr{L}_k(X, Y)$:

$$D^k f(x) = \sum_{j=0}^{\infty} D^k P_j(x) = \sum_{j=k}^{\infty} D^k P_j(x).$$

This follows in the classical way from the fact that the series converges uniformly on every closed ball. This latter fact follows from the relation

$$D^k P_j(x)(h_1, \dots, h_k) = j(j-1)\cdots(j-k+1)B_j(x, \dots, x, h_1, \dots, h_k),$$

which implies

$$||D^k P_j(x)|| \leqslant j(j-1)\cdots(j-k+1)c_j||x||^{j-k}. \tag{7.1}$$

Finally, we will use the notation $M(h, R)$ from the one-dimensional theory for the maximum of the function h on the closed disc with radius R.

Now we can state our convergence result, which is a generalization of Gelfond's theorem about the convergence of Lagrange interpolants to entire functions in $\mathbb{C}$, see Theorem 3 of Chapter II Part 3 in [7].

Theorem 7.1. *Let $p = (p_0, p_1, p_2, \dots)$ be a sequence of vectors in X, let $p^j = (p_0, \dots, p_j)$, and set $r_k := ||p_k||$. Assume that*

(1) *$f : X \to Y$ is an entire power series,*
(2) *the sequence $(r_0, r_1, r_2, \dots)$ is increasing and $r_k \to \infty$,*
(3) *the counting function N, defined by $N(t) = k$ if $r_k \leqslant t$ and $r_{k+1} > t$, satisfies the condition*

$$\log M(\hat{f}, u) \leqslant \lambda N(\theta u),$$

for large enough u, *where*

$$0 < \lambda < \log \frac{1-\theta}{\theta}$$

and $0 < \theta < 1/2$.

Then the sequence $(K_{p^k} f)$ *of Kergin interpolants converges to* f *uniformly on every bounded ball of* X.

Proof. Every bounded ball in X is contained in some closed ball centered at 0 and of radius R, with R different from every r_k, $k \in \mathbb{N}$. We fix such a radius R and prove that $\sup_{||x|| \leqslant R} ||f - K_{p^k} f||$ converges to 0 as $k \to \infty$.

Step 1: An error formula for f. For $f(x) = \sum_{j=0}^{\infty} P_j(x)$, we let $s_k(x)$ denote the partial sum $\sum_{j=0}^{k} P_j(x)$. Since the Kergin operator is a linear projector, we get

$$\begin{aligned} f(x) - K_{p^k} f(x) &= (f - s_k)(x) - K_{p^k}(f - s_k)(x) \\ &= \sum_{j=k+1}^{\infty} P_j(x) - K_{p^k}\left(\sum_{j=k+1}^{\infty} P_j(\cdot)\right)(x). \end{aligned}$$

Now we may use the continuity of the Kergin operator to obtain

$$f(x) - K_{p^k} f(x) = \sum_{j=k+1}^{\infty} (P_j - K_{p^k} P_j)(x).$$

It follows that

$$||f(x) - K_{p^k} f(x)|| \leqslant \sum_{j=k+1}^{\infty} ||P_j(x) - K_{p^k} P_j(x)||. \tag{7.2}$$

Step 2: An error formula for P_j. By the remainder formula in Theorem 6.1, we have

$$\begin{aligned} P_j(x) - K_{p^k} P_j(x) &= \int_{[p_0, p_1, \ldots, p_k, x]} D_{x-p_k} \cdots D_{x-p_0} P_j \\ &= \int_{[p_0, p_1, \ldots, p_k, x]} D^{k+1} P_j(\cdot)(x - p_0, \ldots, x - p_k)\, ds_1 \cdots ds_{k+1}. \end{aligned}$$

Let us estimate this expression. Define, for $s \in S_{k+1}$,

$$\begin{aligned} y_{p^k,x}(s) &:= p_0 + \sum_{i=1}^{k} s_i(p_i - p_0) + s_{k+1}(x - p_0) \\ &= s_0 p_0 + s_1 p_1 + \cdots + s_k p_k + s_{k+1} x, \end{aligned}$$

where $s_0 = 1 - \sum_{i=1}^{k+1} s_i$. By the definition of the simplex, all of the s_i are non-negative numbers. This implies that for $||x|| \leqslant R$, we have

$$\begin{aligned} ||y_{p^k,x}(s)|| &\leqslant s_0||p_0|| + s_1||p_1|| + \cdots + s_k||p_k|| + s_{k+1}||x|| \\ &\leqslant s_0 r_0 + s_1 r_1 + \cdots + s_k r_k + s_{k+1} R \\ &=: y_{r^k,R}(s), \end{aligned}$$

where $r^k = (r_0, \ldots, r_k)$. From (7.1), we deduce that

$$\begin{aligned} &||D_{x-p_k} \cdots D_{x-p_0} P_j(y_{p^k,x})|| \\ &\quad \leqslant j(j-1)\ldots(j-k+1)c_j ||y_{p^k,x}||^{j-k-1} ||x-p_0|| \cdots ||x-p_k|| \\ &\quad \leqslant j(j-1)\ldots(j-k+1)c_j y_{r^k,R}^{j-k-1}(R+r_0)\cdots(R+r_k). \end{aligned}$$

Integrating this over the simplex, we obtain, for $||x|| \leqslant R$,

$$\begin{aligned} &\left|\left| \int_{[p_0,p_1,\ldots,p_k,x]} D_{x-p_k} \cdots D_{x-p_0} P_j \right|\right| \\ &\quad \leqslant \frac{j!c_j}{(j-k-1)!} \int_{[r_0,\ldots,r_k,R]} (\cdot)^{j-k-1}(R+r_0)\cdots(R+r_k). \end{aligned}$$

On the right-hand side, we recognize an expression from a classical one-variable remainder formula, namely:

$$\begin{aligned} &\frac{j!c_j}{(j-k-1)!} \int_{[r_0,\ldots,r_k,R]} (\cdot)^{j-k-1}(R+r_0)\cdots(R+r_k) \\ &\quad = \frac{(R+r_0)\cdots(R+r_k)}{(R-r_0)\cdots(R-r_k)} \int_{[r_0,\ldots,r_k,R]} D_{R-r_0} \cdots D_{R-r_k} c_j(\cdot)^j \\ &\quad = \frac{(R+r_0)\cdots(R+r_k)}{(R-r_0)\cdots(R-r_k)} [c_j R^j - K_{r^k}(c_j(\cdot)^j)(R)]. \end{aligned}$$

Note that the right-hand side is well defined, because the radius R has been taken to be distinct from every r_i. Consequently, we have for $||x|| \leqslant R$ and $j > k$,

$$||P_j(x) - K_{p^k} P_j(x)|| \leqslant \left| \frac{(R+r_0)\cdots(R+r_k)}{(R-r_0)\cdots(R-r_k)} [c_j R^j - K_{r^k}(c_j(\cdot)^j)(R)] \right|. \tag{7.3}$$

Step 3: *Summing up*. Now we will put all of these inequalities together. From (7.2) and (7.3) it follows that, for $||x|| \leqslant R$, we have

$$\begin{aligned} ||f(x) - K_{p^k} f(x)|| &\leqslant \left| \frac{(R+r_0)\cdots(R+r_k)}{(R-r_0)\cdots(R-r_k)} \sum_{j=k+1}^{\infty} [c_j R^j - K_{r^k}(c_j(\cdot)^j)(R)] \right| \\ &= \left| \frac{(R+r_0)\cdots(R+r_k)}{(R-r_0)\cdots(R-r_k)} [\hat{f}(R) - K_{r^k}\hat{f}(R)] \right|. \end{aligned} \tag{7.4}$$

To justify the last line we need to use the continuity of the one-variable Kergin (i.e. Lagrange) operator. Now we have finally proven that

$$\sup_{\|x\| \leqslant R} \|f(x) - K_{p^k} f(x)\| \leqslant \prod_{j=0}^{k} \left| \frac{R + r_j}{R - r_j} \right| |\hat{f}(R) - K_{r^k} \hat{f}(R)|.$$

It remains to show that the right-hand term converges to 0 as $k \to \infty$, and this is a routine one-variable problem. Except for the fact that we have to deal with the parasitic first factor, Gelfond's classical proof works here. By assumption, there is a number η such that

$$\frac{1}{\theta} > \eta > e^{\lambda} + 1,$$

and for fixed such η there is an $\varepsilon > 0$ such that

$$\log(\eta - 1) - \lambda - 2\varepsilon > 0.$$

Now an application of Hermite's classical remainder formula (see e.g. [7, Formula 123 of Chapter II Part 3]) with the circle of radius ηr_k centered at the origin as the contour of integration, leads to

$$|\hat{f}(R) - K_{r^k} \hat{f}(R)| \leqslant \frac{\eta r_k}{|\eta r_k - R|} \frac{(1 + R/r_k)^{k+1}}{(\eta - 1)^{k+1}} M(\hat{f}, \eta r_k).$$

By assumption we have for k large, say $k > k_0$,

$$\log M(\hat{f}, \eta r_k) \leqslant \lambda N(\theta \eta r_k) \leqslant \lambda k.$$

Since $R/r_k \to 0$ as $k \to \infty$, one can find a constant C such that

$$\left| \frac{(R + r_0) \cdots (R + r_k)}{(R - r_0) \cdots (R - r_k)} \right| \leqslant C e^{k\varepsilon}$$

and

$$\left(1 + \frac{R}{r_k}\right)^{k+1} \leqslant C e^{k\varepsilon}.$$

Putting it all together: for $k > k_0$ we get that

$$\prod_{j=0}^{k} \left| \frac{R + r_j}{R - r_j} \right| |\hat{f}(R) - K_{r^k} \hat{f}(R)| \leqslant C \exp(-k(\log(\eta - 1) - \lambda - 2\varepsilon)),$$

and the conclusion follows, since the factor multiplying $-k$ is positive. □

Remark 7.2. It is perhaps worth pointing out that one special case in which Theorem 7.1 is applicable is when $X = Y = E$, a complex Banach algebra with unity, and

$f: E \to E$ is an mapping of the form

$$f(x) = \sum_{j=0}^{\infty} a_j x^j,$$

where the a_j are elements of E and the radius of convergence is infinite (of course, $a_j x^j$ is to be interpreted as $a_j \cdot x \cdot x \cdot \ldots \cdot x$, with the $\cdot$ denoting the multiplication in E and the x occurring j times). A concrete example is $E = M_n(\mathbb{C})$, the algebra of square matrices with complex coefficients, and $f(x) = \exp(x)$, the exponential of a matrix.

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